ON A CERTAIN CLASS OF SELF-PROJECTIVE SURFACES*

BY

E. J. WILCZYNSKI

Introduction.

Let P be a point of a surface S, and let R be a non-vanishing two-dimensional portion of S, finite or infinite in extent, such that P is one of its interior points and such that the homogeneous coördinates $x_1, \cdots, x_4$ of any point of R may be expressed as convergent series of positive integral powers of two parameters u and v. Moreover, let the two asymptotic tangents of the surface at P be distinct, and let us assume further that neither of these asymptotic tangents has contact of higher than the second order with the surface at P. The tetrahedron of reference may then be chosen in such a way that the surface may be represented by a canonical development of the form

$$z = xy + \frac{1}{6}(x^3 + y^3) + \frac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where the non-homogeneous coördinates are defined by the equations

$$x = \frac{x_1}{x_4}, \qquad y = \frac{x_2}{x_4}, \qquad z = \frac{x_3}{x_4},$$

and where I, J and all further coefficients of the development are absolute differential invariants of the surface.†

It is the purpose of this paper to study those surfaces for which the two invariants I and J are equal to zero, at all points which satisfy the conditions formulated above.

§ 1. *Partial differential equations of the surfaces referred to their asymptotic curves*

If the curves $u =$ const. and $v =$ const. of the surface

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4)$$

* Presented to the Society, September 11, 1912.

† E. J. WILCZYNSKI, *Projective Differential Geometry of Curved Surfaces* (*Second Memoir*). These Transactions, vol. 9 (1908), p. 103. We shall hereafter refer to this paper as Second Memoir.

are asymptotic curves, there will exist a completely integrable system of partial differential equations*

$$(1)\qquad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

of which $y^{(1)}, \cdots, y^{(4)}$ are four linearly independent solutions. The integrability conditions of this system are†

$$(2)\qquad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0. \end{aligned}$$

Let us put

$$(3)\qquad A = a'b^2, \qquad B = a'^2 b.$$

Then the invariants I and J, which occur in the canonical development, have the following values‡

$$I = \frac{B_u}{4B\sqrt[3]{A}}, \qquad J = \frac{A_v}{4A\sqrt[3]{B}},$$

where, under our assumptions, A and B are both different from zero. For if either a' or b were equal to zero identically the surface S would be a ruled surface, and one of the asymptotic tangents of such a point would have higher than second order contact with the surface. But this is a case which we have explicitly excluded from consideration.

If I and J are identically equal to zero, we must have

$$A = U, \qquad B = V,$$

where U and V are non-vanishing functions of the single variables u and v respectively. Now A and B are relative invariants of the system of differential equations (1). If we introduce new independent variables $\bar{u}$ and $\bar{v}$ by means of the transformation

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

the new values $\bar{A}$ and $\bar{B}$ of these invariants will be connected with their original values A and B by the following relations§,

$$(4)\qquad \bar{A} = \frac{A}{\alpha_u^3}, \qquad \bar{B} = \frac{B}{\beta_v^3}.$$

* *First Memoir*, these Transactions, vol. 8 (1907), p. 246.

† *Loc. cit.*, p. 247.

‡ *Second Memoir*, p. 103.

§ *First Memoir*, p. 249.

Since in our case A is a function of u alone, and B a function of v alone, the functions $\alpha(u)$ and $\beta(v)$ may be chosen in such a way as to reduce $\bar{A}$ and $\bar{B}$ to unity. Let us assume that this reduction has been made, so that

$$A = B = 1 .$$

We conclude from (3) that

$$a' = b = 1 \quad \text{or} \quad \omega \quad \text{or} \quad \omega^2 ,$$

where ω is an imaginary cube root of unity. But we may, without any restriction of generality, assume

$$a' = b = 1 .$$

For, as we see from (4), the conditions $A = B = 1$ will not be disturbed by a transformation of the independent variables for which $\alpha_u^3 = \beta_v^3 = 1$. Now a' and b are themselves relative invariants and, if their common value is ω or ω^2, it is possible to find a linear transformation of the independent variables reducing both of them to unity.

If we substitute $a' = b = 1$ into the integrability conditions (2), they reduce to

$$g_u = 0, \qquad f_v = 0, \qquad -f_u + g_v = 0,$$

so that

$$f = U, \qquad g = V,$$

U and V being functions of the single variables u and v respectively, and

$$U_u = V_v .$$

Since the left member is a function of u alone, and the right member depends only upon v, their common value must be a constant c_0, so that finally

$$f = c_0 u + c_1, \qquad g = c_0 v + c_2 .$$

We have found the following theorem. *The integral surfaces of the completely integrable system of partial differential equations*

$$\begin{aligned} y_{uu} + 2y_v + (c_0 u + c_1) y &= 0, \\ y_{vv} + 2y_u + (c_0 v + c_2) y &= 0 \end{aligned} \tag{5}$$

are the only surfaces at all of whose points the invariants I and J are equal to zero.

These surfaces have the following characteristic property. *The canonical cubic of every point P of such a surface S has fourth order contact with S at P.**

* In these and all similar statements, the locution "every point" is intended to mean every point which satisfies the conditions formulated at the beginning of the introduction. For the definition of the "canonical cubic," cf. *Second Memoir*, p. 108.

§ 2. *Determination of the surfaces for the case when $c_0 = 0$ and when the characteristic equations have distinct roots.**

If $c_0 = 0$, system (5) reduces to

$$
(6) \qquad \begin{aligned} y_{uu} + 2y_v + c_1 y = 0, \\ y_{vv} + 2y_u + c_2 y = 0, \end{aligned}
$$

and may be integrated by elementary functions. In fact we may put

$$
y = e^{\alpha u + \beta v},
$$

where α and β are constants which satisfy the *characteristic* equations

$$
(7) \qquad \begin{aligned} \alpha^2 + 2\beta + c_1 = 0, \\ \beta^2 + 2\alpha + c_2 = 0. \end{aligned}
$$

Elimination of β gives

$$
(8) \qquad \alpha^4 + 2c_1 \alpha^2 + 8\alpha + c_1^2 + 4c_2 = 0, \qquad \beta = -\tfrac{1}{2}(c_1 + \alpha^2),
$$

and similarly we find

$$
(9) \qquad \beta^4 + 2c_2 \beta^2 + 8\beta + c_2^2 + 4c_1 = 0, \qquad \alpha = -\tfrac{1}{2}(c_2 + \beta^2).
$$

The discriminants of (8) and (9), except for a numerical factor, are both equal to

$$
(10) \qquad \Delta = c_1^2 c_2^2 + 4(c_1^3 + c_2^3) + 18 c_1 c_2 - 27.
$$

We shall first assume that Δ is not equal to zero, so that the four roots $\alpha_1, \cdots, \alpha_4$ of (8) and $\beta_1, \cdots, \beta_4$ of (9) are distinct. We shall then have four solutions of the form

$$
(11) \qquad y_k = e^{\alpha_k u + \beta_k v} \qquad (k = 1, 2, 3, 4),
$$

which are linearly independent. The surface may therefore be represented by equations (11), the curves $u =$ const. and $v =$ const. being its asymptotic lines.

We proceed to find the equation of the surface by eliminating u and v. We find

$$
(12) \qquad \log \frac{y_2^{\alpha_3 - \alpha_1} y_1^{\alpha_2 - \alpha_3}}{y_3^{\alpha_2 - \alpha_1}} = Av, \qquad \log \frac{y_2^{\alpha_4 - \alpha_1} y_1^{\alpha_2 - \alpha_4}}{y_4^{\alpha_2 - \alpha_1}} = Bv,
$$

where

$$
\begin{aligned}
A = \beta_1(\alpha_2 - \alpha_3) + \beta_2(\alpha_3 - \alpha_1) + \beta_3(\alpha_1 - \alpha_2) \\
= \tfrac{1}{2}(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)(\alpha_1 - \alpha_2), \\
B = \beta_1(\alpha_2 - \alpha_4) + \beta_2(\alpha_4 - \alpha_1) + \beta_4(\alpha_1 - \alpha_2) \\
= \tfrac{1}{2}(\alpha_2 - \alpha_4)(\alpha_4 - \alpha_1)(\alpha_1 - \alpha_2).
\end{aligned}
$$

---

* Discussion of the cases when $c_0 \neq 0$ will be reserved for a subsequent paper.

Consequently, if we introduce non-homogeneous coördinates by putting

$$X = \frac{y_2}{y_1}, \qquad Y = \frac{y_3}{y_1}, \qquad Z = \frac{y_4}{y_1},$$

and if we use the notation

$$\lambda = \frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1}, \qquad \mu = \frac{\alpha_4 - \alpha_1}{\alpha_2 - \alpha_1},$$

we find

$$(X^\lambda Y^{-1})^{\mu(1-\mu)} = (X^\mu Z^{-1})^{\lambda(1-\lambda)}$$

or

$$X = Y^{\frac{1-\mu}{\lambda(\lambda-\mu)}} Z^{-\frac{1-\lambda}{\mu(\lambda-\mu)}} = Y^\alpha Z^\beta \tag{13}$$

as the equation of our surface, which is clearly self-projective, being left invariant by the transformations of a two-parameter projective group.

The quantities λ and μ, and also α and β, are clearly functions of c_1 and c_2. The question immediately suggests itself whether λ and μ may be chosen as arbitrary constants or whether they are subject to some restrictions. It is clear in the first place that we must assume

$$\lambda \neq 0, 1, \infty; \quad \mu \neq 0, 1, \infty, \quad \lambda \neq \mu,$$

since the four roots α_1, α_2, α_3, α_4 of (8) are supposed to be distinct. With this understanding we find

$$\alpha_3 = \alpha_1 + \lambda(\alpha_2 - \alpha_1), \quad \alpha_4 = \alpha_1 + \mu(\alpha_2 - \alpha_1), \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0.$$

Of course, one of the roots of (8) may be equal to zero. But, since we are now dealing with the case of distinct roots, there can be only one such root. Let the notation be fixed so that $\alpha_1 \neq 0$, and therefore $\lambda + \mu + 1 \neq 0$. We shall find

$$\alpha_2 = \frac{\lambda + \mu - 3}{\lambda + \mu + 1}\alpha_1, \qquad \alpha_3 = \frac{-3\lambda + \mu + 1}{\lambda + \mu + 1}\alpha_1, \qquad \alpha_4 = \frac{\lambda - 3\mu + 1}{\lambda + \mu + 1}\alpha_1.$$

According to (8) the sum of the products of α_i, α_j, α_k, taken three at a time, must be equal to -8. Thus

$$\begin{aligned}\alpha_1^3[&(\lambda + \mu - 3)(-3\lambda + \mu + 1)(\lambda - 3\mu + 1)\\ &+ (-3\lambda + \mu + 1)(\lambda - 3\mu + 1)(\lambda + \mu + 1)\\ &+ (\lambda - 3\mu + 1)(\lambda + \mu - 3)(\lambda + \mu + 1)\\ &+ (\lambda + \mu - 3)(-3\lambda + \mu + 1)(\lambda + \mu + 1)]\\ &= -8(\lambda + \mu + 1)^3.\end{aligned} \tag{14}$$

If the coefficient of α_1^3 in this equation is not equal to zero, we can therefore express $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as functions of λ and μ. The two remaining symmetric functions of the roots of (8) will then give us $2c_1$ and $c_1^2 + 4c_2$, so that c_1 and c_2 are expressible as algebraic functions of λ and μ. In other words; if λ and μ are any two numbers which do not cause the above expression to vanish, the surface (13) will be an integral surface of a system of differential equations of form (6).

The coefficient of α_1^3 in (14) may be reduced to -8 times the expression

$$\lambda^3 + \mu^3 - \lambda^2\mu - \lambda\mu^2 - \lambda^2 - \mu^2 + 2\lambda\mu - \lambda - \mu + 1$$

and this vanishes if and only if the anharmonic curve $v =$ const., represented by equations (12), belongs to a linear complex.* But surfaces whose asymptotic lines belong to linear complexes are indeed excluded from our class of surfaces. In fact the differential equation of the fourth order characteristic of the asymptotic lines $v =$ const. is found to be

$$y_{uuuu} + 2c_1 y_{uu} + 8y_u + (c_1^2 + 4c_2)\, y = 0,$$

whose invariant of weight three is found to be independent of c_1 and c_2, namely equal to 2. In no case then, can this invariant vanish, i. e., *in no case will an asymptotic curve of the surface S belong to a linear complex. The asymptotic curves of S are clearly anharmonic curves whose invariants are subject to no other restriction than this, the curves of each family being projectively equivalent to each other, but in general (if $c_1 \neq c_2$) not equivalent to those of the other family.* Moreover, the invariants of each family determine those of the other and of the surface.†

Thus, *any surface of the form* (13) *may be regarded as an integral surface of system* (6), *i. e., as a surface for which I and J are everywhere equal to zero, provided that λ and μ are different from each other, from* 0, 1, *or* ∞, *and do not satisfy the condition*

$$\lambda^3 + \mu^3 - \lambda^2\mu - \lambda\mu^2 - \lambda^2 - \mu^2 + 2\lambda\mu - \lambda - \mu + 1 = 0. \tag{15}$$

The corresponding restrictions on the exponents α and β, which enter into the equation of the surface, may be found as follows. We have

$$\alpha = \frac{1-\mu}{\lambda(\lambda-\mu)}, \qquad \beta = -\frac{1-\lambda}{\mu(\lambda-\mu)},$$

whence

$$-\alpha(\alpha+\beta)\lambda^2 + 2\alpha\lambda + \beta - 1 = 0, \qquad \mu = \frac{1-\alpha\lambda^2}{1-\alpha\lambda}, \tag{16a}$$

* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, B. G. Teubner, Leipzig, 1906, p. 281. In the notation there used $\theta_3 = 0$ is the condition for a curve belonging to a linear complex. We shall refer to this book as Proj. Diff. Geom.

† *Loc. cit.*, p. 281.

and

$$(16b) \qquad -\beta(\alpha+\beta)\mu^2 + 2\beta\mu + \alpha - 1 = 0, \qquad \lambda = \frac{1-\beta\mu^2}{1-\beta\mu}.$$

Since α and β are supposed to be finite, and λ and μ must not be equal to $0, 1$ or ∞, we must have

$$\alpha \neq 1, \quad \beta \neq 1, \quad -\alpha(\alpha+\beta) + 2\alpha + \beta - 1 \neq 0, \quad \alpha(\alpha+\beta) \neq 0,$$
$$-\beta(\alpha+\beta) + 2\beta + \alpha - 1 \neq 0, \quad \beta(\alpha+\beta) \neq 0.$$

But, if $\lambda = 1$, μ will also be equal to unity. Therefore the third and fourth inequalities must hold simultaneously. But we may write these inequalities as follows

$$(\alpha-1)(\alpha+\beta-1) \neq 0, \qquad (\beta-1)(\alpha+\beta-1) \neq 0,$$

whence $\alpha+\beta-1 \neq 0$. We must also exclude the possibility $\lambda = \mu$. But, if $\lambda = \mu$, we find

$$\lambda - \alpha\lambda^2 = 1 - \alpha\lambda^2,$$

so that λ is either equal to one or infinity, which cases have already been discussed. We see therefore that α and β are subject to the inequalities

$$(17) \qquad \alpha \neq 0, \quad \alpha \neq 1, \quad \beta \neq 0, \quad \beta \neq 1, \quad \alpha+\beta \neq 0, \quad \alpha+\beta \neq 1.$$

From (16) we find

$$\lambda = \frac{\alpha\beta \pm \beta t}{\alpha\beta(\alpha+\beta)}, \qquad \mu = \frac{\alpha\beta \pm \alpha t}{\alpha\beta(\alpha+\beta)}, \qquad t^2 = \alpha\beta(\alpha+\beta-1),$$

and therefore

$$\lambda + \mu = \frac{2\alpha\beta \pm (\beta-\alpha)t}{\alpha\beta(\alpha+\beta)}, \qquad \lambda\mu = \frac{\alpha\beta(2-\alpha-\beta) \pm (\beta-\alpha)t}{\alpha\beta(\alpha+\beta)^2},$$

$$(\lambda-\mu)^2 = \frac{(\alpha+\beta-1)^2}{\alpha\beta},$$

$$\lambda\mu(\lambda+\mu) = \frac{2\alpha\beta(2-\alpha-\beta) + (\beta-\alpha)^2(\alpha+\beta-1) \pm (\beta-\alpha)(4-\alpha-\beta)t}{\alpha\beta(\alpha+\beta)^3},$$

$$(\lambda+\mu)^3 = \frac{\alpha\beta[8\alpha\beta + 6(\beta-\alpha)^2(\alpha+\beta-1)] \pm (\beta-\alpha)[12\alpha\beta + (\beta-\alpha)^2(\alpha+\beta-1)]t}{\alpha^2\beta^2(\alpha+\beta)^3}.$$

Combined with these equations, the final restriction that λ and μ shall not satisfy (15) gives rise to the inequality

$$(18) \qquad \alpha\beta[(\alpha+\beta)(\alpha+\beta-2)(\alpha-1)(\beta-1) - 4\alpha\beta]^2 \neq (\alpha+\beta)^2(\alpha-\beta)^2(\alpha-1)^2(\beta-1)^2(\alpha+\beta-1).$$

If, then, the discriminant Δ *is not equal to zero, the integral surfaces of* (6) *are projective transformations of the surfaces*

$$X = Y^{\alpha} Z^{\beta},$$

in which the constant exponents α *and* β *are arbitrary except for the restrictions implied by the inequalities* (17) *and* (18). *The surfaces thus excluded either are ruled or have asymptotic lines belonging to linear complexes.*

We shall speak of the surfaces obtained so far as surfaces of the first type. We shall say that a plane anharmonic curve is of the first type, if the one-parameter group of collineations, which leaves it invariant, has an invariant triangle. It is evident that *all plane sections of a surface of the first type are plane anharmonic curves of the first type.*

§ 3. *Discussion of the case when* $c_0 = 0$, *and when two or more roots of the characteristic equations coincide.*

Let α_3 and α_4 be equal and let the other roots of (8) be distinct and different from α_3. Of course, the discriminant Δ must be equal to zero, so that

$$c_1^2 c_2^2 + 4(c_1^3 + c_2^3) + 18 c_1 c_2 - 27 = 0. \tag{19}$$

The three functions

$$y_k = e^{\alpha_k u + \beta_k v} \qquad (k = 1, 2, 3), \tag{20}$$

will still be linearly independent. In accordance with well-known principles, we may choose as a fourth independent solution the function

$$\left[\frac{\partial e^{\alpha u + \beta v}}{\partial \alpha}\right]_{\alpha = \alpha_3}$$

Since we have

$$\beta = -\tfrac{1}{2}(c_1 + \alpha^2),$$

we find in this way

$$y_4 = (u - \alpha_3 v)\, e^{\alpha_3 u + \beta_3 v} \tag{21}$$

Introducing non-homogeneous coördinates, we find

$$X = \frac{y_1}{y_3} = e^{(\alpha_1 - \alpha_3)u + (\beta_1 - \beta_3)v}, \qquad Y = \frac{y_2}{y_3} = e^{(\alpha_2 - \alpha_3)u + (\beta_2 - \beta_3)v},$$
$$Z = \frac{y_4}{y_3} = u - \alpha_3 v. \tag{22}$$

In order to be able to eliminate u and v, it is convenient to express α_1, α_2, β_1, β_2 as functions of α_3. We find from equations (8) that

$$\alpha_1 + \alpha_2 = -2\alpha_3, \qquad \alpha_1 \alpha_2 = 2c_1 + 3\alpha_3^2, \tag{23}$$

so that α_1 and α_2 are the two roots of the quadratic

$$\alpha^2 + 2\alpha_3\,\alpha + 2c_1 + 3\alpha_3^2 = 0.$$

But we find further from (8) and (23) that

$$2\alpha_3\,(2c_1 + 3\alpha_3^2) - 2\alpha_3^3 = -8, \qquad (2c_1 + 3\alpha_3^2)\,\alpha_3^2 = c_1^2 + 4c_2,$$

whence

$$\alpha_3^3 + c_1\,\alpha_3 + 2 = 0, \quad 3\alpha_3^4 + 2c_1\,\alpha_3^2 - (c_1^2 + 4c_2) = 0.$$

By means of the first of these equations, the quadratic whose roots are α_1 and α_2 becomes

$$\alpha^2 + 2\alpha_3\,\alpha + \alpha_3^2 - \frac{4}{\alpha_3} = 0,$$

so that we may write

$$\alpha_1 = -\alpha_3 + \frac{2}{\sqrt{\alpha_3}}, \qquad \alpha_2 = -\alpha_3 - \frac{2}{\sqrt{\alpha_3}},$$

where α_3 cannot be equal to zero, since (8) clearly never has zero as a double root. We find

$$\beta_1 = -\tfrac{1}{2}\,(c_1 + \alpha_1^2) = \tfrac{1}{2}\,(\alpha_3^2 - \alpha_1^2) + \frac{1}{\alpha_3},$$

$$\beta_2 = -\tfrac{1}{2}\,(c_1 + \alpha_2^2) = \tfrac{1}{2}\,(\alpha_3^2 - \alpha_2^2) + \frac{1}{\alpha_3},$$

$$\beta_3 = -\tfrac{1}{2}\,(c_1 + \alpha_3^2) = \frac{1}{\alpha_3},$$

so that

$$(24)\qquad \begin{aligned} \alpha_1 - \alpha_3 &= 2\left(-\alpha_3 + \frac{1}{\sqrt{\alpha_3}}\right), \qquad \beta_1 - \beta_3 = 2\left(-\frac{1}{\alpha_3} + \sqrt{\alpha_3}\right), \\ \alpha_2 - \alpha_3 &= 2\left(-\alpha_3 - \frac{1}{\sqrt{\alpha_3}}\right), \qquad \beta_2 - \beta_3 = 2\left(-\frac{1}{\alpha_3} - \sqrt{\alpha_3}\right), \end{aligned}$$

and

$$(\alpha_1 - \alpha_3)\,(\beta_2 - \beta_3) - (\alpha_2 - \alpha_3)\,(\beta_1 - \beta_3) = 8\alpha_3^{-3/2}\,(\alpha_3^3 - 1).$$

This latter expression is different from zero. For, if α_3 were equal to a cube-root of unity, either $\alpha_1 - \alpha_3$ or $\alpha_2 - \alpha_3$ would vanish, i. e., α_3 would be a triple root of the characteristic equation, contrary to our assumption. We find, therefore, from (22)

$$4\alpha_3^{-3/2}(\alpha_3^3 - 1)\,u = -\left(\frac{1}{\alpha_3} + \sqrt{\alpha_3}\right)\log X - \left(-\frac{1}{\alpha_3} + \sqrt{\alpha_3}\right)\log Y,$$

$$4\alpha_3^{-3/2}(\alpha_3^3 - 1)\,v = \left(\alpha_3 + \frac{1}{\sqrt{\alpha_3}}\right)\log X + \left(-\alpha_3 + \frac{1}{\sqrt{\alpha_3}}\right)\log Y,$$

and consequently

$$4\alpha_3^{-3/2}(\alpha_3^3 - 1)\,Z = -\frac{(\alpha_3^{3/2}+1)^2}{\alpha_3}\log X + \frac{(\alpha_3^{3/2}-1)^2}{\alpha_3}\log Y,$$

or

$$\frac{4}{\sqrt{\alpha_3}}Z = -\frac{\alpha_3^{3/2}+1}{\alpha_3^{3/2}-1}\log X + \frac{\alpha_3^{3/2}-1}{\alpha_3^{3/2}+1}\log Y,$$

a surface which is projectively equivalent to

$$Z = \log\frac{Y^k}{X^{1/k}}, \tag{25}$$

where k is any finite non-vanishing constant. *We shall speak of* (25) *as a surface of the second type.* Clearly, *only the plane sections* $Z = const.$ *are anharmonic curves of the first type. All other plane sections are anharmonic curves of the second type.*

Consider now the case when the characteristic equation has three equal roots, say

$$\alpha_2 = \alpha_3 = \alpha_4 = \alpha.$$

We shall then have, according to (8),

$$\alpha_1 = -3\alpha, \quad c_1 = -3\alpha^2, \quad \alpha^3 = 1, \quad c_2 = -3\alpha.$$

Our system of differential equations (6) assumes the form

$$y_{uu} + 2y_v - 3\alpha^2 y = 0,$$

$$y_{vv} + 2y_u - 3\alpha y = 0,$$

where α is a cube-root of unity.

By a transformation of the form

$$\bar{u} = \lambda u, \qquad \bar{v} = \mu v,$$

λ and μ being constants, this system becomes

$$\frac{\partial^2 y}{\partial \bar{u}^2} + 2\frac{\mu}{\lambda^2}\frac{\partial y}{\partial \bar{v}} - 3\frac{\alpha^2}{\lambda^2}y = 0,$$

$$\frac{\partial^2 y}{\partial \bar{v}^2} + 2\frac{\lambda}{\mu^2}\frac{\partial y}{\partial \bar{u}} - 3\frac{\alpha}{\mu^2}y = 0,$$

and this is of the same form as the original system if

$$\lambda^2 = \mu, \qquad \mu^2 = \lambda,$$

i. e., if

$$\lambda = 1,\ \omega,\ \omega^2; \qquad \mu = 1,\ \omega^2,\ \omega,$$

where ω is an imaginary cube root of unity.

Thus, if $\alpha = \omega$, the above transformation for $\lambda = \omega$ will give rise to a new system of the same form for which $\alpha = 1$. Similarly if $\alpha = \omega^2$. We may assume therefore, without any loss of generality, that $\alpha = 1$, so that we have to deal only with the system

$$y_{uu} + 2y_v - 3y = 0, \qquad y_{vv} + 2y_u - 3y = 0. \tag{26}$$

Equation (8) reduces to

$$(\alpha - 1)^3 (\alpha + 3) = 0,$$

so that

$$\alpha_1 = -3, \qquad \alpha_2 = \alpha_3 = \alpha_4 = 1,$$

$$\beta_1 = -3, \qquad \beta_2 = \beta_3 = \beta_4 = 1.$$

We find the following solutions

$$y_1 = e^{-3(u+v)}, \qquad y_2 = e^{u+v}, \qquad y_3 = (u - v)e^{u+v}, \tag{27}$$

$$y_4 = [2(u - v)^2 - (u + v)] e^{u+v}.$$

Therefore

$$X = \frac{y_3}{y_2} = u - v; \qquad Y = \frac{y_4}{y_2} = 2(u - v)^2 - (u + v), \tag{28}$$

$$Z = \frac{y_1}{y_2} = e^{-4(u+v)},$$

whence

$$Z = e^{4(Y-2X^2)}, \tag{29}$$

the equation of our surfaces of the third type.

If equation (8) had four equal roots, all of these roots would be equal to zero, since their sum must vanish. But this is clearly impossible, since the sum of their products taken three at a time must be equal to -8. If (8) had two pairs of equal roots we should again encounter the same contradiction. Therefore these two cases are impossible, and *the integral surfaces of system* (6) *must always belong to one of the three types which we have determined.*

§ 4. *General properties of the surfaces of the problem.*

The curves $u =$ const. and $v =$ const. are the two families of asymptotic lines of our surface S. The tangents of the curves of the second family, constructed at all of the points of a curve $u =$ const. of the first family, generate a ruled surface R_1, the osculating ruled surface of the first kind.* If we write

$$z = y_u,$$

* *Second Memoir*, p. 81.

the differential equations of this ruled surface will be

$$y_{vv} + p_{11} y_v + p_{12} z_v + q_{11} y + q_{12} z = 0,$$

$$z_{vv} + p_{21} y_v + p_{22} z_v + q_{21} y + q_{22} z = 0,$$

where *

$$p_{11} = p_{12} = 0, \quad q_{11} = c_2, \quad q_{12} = 2,$$

$$p_{21} = -4, \quad p_{22} = 0, \quad q_{21} = -2c_1, \quad q_{22} = c_2,$$

so that †

$$u_{11} - u_{22} = 0, \quad u_{12} = -8, \quad u_{21} = 8c_1,$$

$$v_{11} - v_{22} = -64, \quad v_{12} = 0, \quad v_{21} = 0,$$

$$w_{11} - w_{22} = 0, \quad w_{12} = 0, \quad w_{21} = 256,$$

$$\theta = (u_{11} - u_{22})^2 + 4u_{12} u_{21} = -2^8 c_1, \quad \Delta = -2^{17}.$$

Since $u_{11} - u_{22} = 0$, *the curves* C_y *and* C_z *on* R_1 *are harmonic conjugates of each other with respect to the two branches of the flecnode curve, which two branches coincide if and only if* $c_1 = 0$.

In general the two branches of the flecnode curve of R_1 are found‡ by factoring the covariant C. In our case, except for a numerical factor, this covariant is equal to $c_1 y^2 + z^2$, so that we find the following parametric expressions for the coördinates of the points of the two branches of the flecnode curve of R_1;

$$x_k = z_k \pm \sqrt{-c_1}\, y_k \qquad (k = 1, 2, 3, 4).$$

For a surface of the first type this becomes

$$x_k = (\alpha_k \pm \sqrt{-c_1})\, e^{\alpha_k u + \beta_k v} = (\alpha_k \pm \sqrt{-c_1})\, y_k.$$

Thus; *each of the branches of the flecnode curve of* R_1 *is a projective transformation of the corresponding asymptotic curve of* S; *their locus is a surface of two sheets, each projectively equivalent to* S *itself*; *except when one of the roots* $\alpha_1, \cdots \alpha_4$ *of the characteristic equation satisfy the condition* $\alpha^2 + c_1 = 0$, *in which case one of the branches of the flecnode curve becomes a plane curve, and the corresponding sheet of the surface which is their locus degenerates into a plane.*

Since our surface S is not a ruled surface, and since no plane curve which is not a straight line can be an asymptotic line of a surface, we see that *a branch of a flecnode curve on one of the osculating ruled surfaces* R_1 *can never become a plane curve except under the condition just stated.* This may be verified analytically by calculating the Wronskian of $x_1, \cdots x_4$ with respect to v.

* *Loc. cit.*, p. 81.

† *Proj. Diff. Geom.*, pp. 96, 99, 101, 102, 104.

‡ *Proj. Diff. Geom.*, p. 150.

If one of the quantities $\alpha_1, \cdots \alpha_4$ satisfies the condition

$$\alpha_i^2 + c_1 = 0,$$

β_i will vanish, and according to (9) this implies $c_2^2 + 4c_1 = 0$. Thus, *one of the branches of the flecnode curve on the osculating ruled surfaces R_1 of S will be a plane curve, if and only if*

$$(30)_1 \qquad c_2^2 + 4c_1 = 0.$$

The corresponding condition for the osculating ruled surfaces R_2 of the second kind is

$$(30)_1 \qquad c_1^2 + 4c_2 = 0.$$

If both conditions are satisfied we must have for c_1, c_2 one of the four sets of values

$$c_1 = 0, \quad -4, \quad -4\omega, \quad -4\omega^2,$$

$$c_2 = 0, \quad -4, \quad -4\omega^2, \quad -4\omega,$$

where ω is a complex cube root of unity. As in the preceding paragraph these cases may be reduced to two, viz.:

$$c_1 = c_2 = 0 \quad \text{or} \quad c_1 = c_2 = -4.$$

Those surfaces of the first type all of whose osculating ruled surfaces, of both kinds, have a flecnode curve with a plane branch are integral surfaces of a system of form (6) *in which either $c_1 = c_2 = 0$ or $c_1 = c_2 = -4$. The flecnode curves of the osculating ruled surfaces of the surfaces belonging to the first of these two classes, $(c_1 = c_2 = 0)$, coincide entirely with the plane branch. For these surfaces the two branches of the flecnode curve coincide. This is not the case for the surfaces of the second class $(c_1 = c_2 = -4)$.*

These two cases merit closer attention. We have, in the first case, $c_1 = c_2 = 0$ and therefore

$$y_1 = e^{-2(u+v)}, \qquad y_2 = e^{-2(\omega u+\omega^2 v)}, \qquad y_3 = e^{-2(\omega^2 u+\omega v)}, \qquad y_4 = 1,$$

where

$$\omega = -\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3}, \qquad \omega^2 = -\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3},$$

so that

$$y_2 = e^{u+v+i\sqrt{3}(-u+v)}, \qquad y_3 = e^{u+v+i\sqrt{3}(u-v)}.$$

Let us introduce non-homogeneous coördinates by means of the projective transformation

$$\xi = \frac{y_2 + y_3}{2y_1}, \qquad \eta = \frac{y_3 - y_2}{2iy_1}, \qquad \zeta = \frac{y_4}{y_1}.$$

We find

$$(31)\qquad \begin{aligned} \xi &= e^{3(u+v)} \cos \sqrt{3}\,(u-v), \\ \eta &= e^{3(u+v)} \sin \sqrt{3}\,(u-v), \\ \zeta &= e^{2(u+v)}, \end{aligned}$$

whence we obtain

$$(32)\qquad \zeta^3 = \xi^2 + \eta^2$$

as the equation of our surface. If we interpret ξ, η, ζ as Cartesian coördinates, as we may, *this is a surface of revolution generated by rotating the semi-cubical parabola*

$$\xi^2 = \zeta^3$$

about the ζ axis.

The asymptotic lines of the surface are obtained by equating either u or v in (31) to a constant. *The projections of these curves on the $\xi\eta$ plane are logarithmic spirals which intersect the radii vectores at a constant angle of* 60°. *They are, therefore, curves all of whose points are coincidence points.**

We know already that the flecnode curves of the osculating ruled surfaces have coincident branches, and are plane curves. It is easy to show that these plane curves coincide with the above two families of logarithmic spirals. More specifically we may express our result as follows.

Every asymptotic line A of the surface

$$(S)\qquad \zeta^3 = \xi^2 + \eta^2$$

has as its projection in the $\xi\eta$ plane a logarithmic spiral intersecting all radii vectores at an angle of 60°. *The flecnode curve of the ruled surface which osculates S along A is a logarithmic spiral obtained from the projection of A by a rotation of* 120° *around the origin.*

A somewhat similar situation exists for surfaces of the second class, for which $c_1 = c_2 = -4$. We find, in this case, the fundamental solutions

$$(33)\qquad y_1 = e^{2v}, \qquad y_2 = e^{2u}, \qquad y_3 = e^{(-1+\sqrt{5})(u+v)}, \qquad y_4 = e^{-(1+\sqrt{5})(u+v)}.$$

If we put

$$X = \frac{y_1}{y_4}, \qquad Y = \frac{y_2}{y_4}, \qquad Z = \frac{y_3}{y_4},$$

we find

$$(34)\qquad Z = (XY)^{5-2\sqrt{5}}$$

as the equation of the surface, with

$$(35)\qquad X = e^{(1+\sqrt{5})u+(3+\sqrt{5})v}, \qquad Y = e^{(3+\sqrt{5})u+(1+\sqrt{5})v}, \qquad Z = e^{2\sqrt{5}(u+v)}$$

as its parametric representation.

* *Proj. Diff. Geom.*, p. 70.

Consider an asymptotic line $v = \text{const.}$ of this surface and the osculating ruled surface R_1 which corresponds to it. The homogeneous coördinates x'_k and x''_k of the points which describe the two branches of the flecnode curve of R_1 will be

$$x'_1 = 2y'_1, \quad x'_2 = 4y_2, \quad x'_3 = (1 + \sqrt{5})\, y_3, \quad x'_4 = -(-1 + \sqrt{5})\, y_4,$$

$$x''_1 = -2y_1, \quad x''_2 = 0, \quad x''_3 = (-3 + \sqrt{5}) y_3, \quad x''_4 = -(3 + \sqrt{5})\, y_4.$$

Thus *the second branch of the flecnode curve is indeed a plane curve in the XZ plane, the non-homogeneous coördinates of whose points are given by*

$$X'' = \frac{2}{3 + \sqrt{5}} X, \qquad Y'' = 0, \qquad Z'' = \frac{3 - \sqrt{5}}{3 + \sqrt{5}} Z.$$

These curves clearly coincide with the projections on the XZ plane of the asymptotic curves $v = \text{const.}$ of the original surface. An analogous relation exists between the plane branches of the flecnode curves of the osculating ruled surfaces R_2, and the projections on the YZ plane of the asymptotic curves $u = \text{const.}$

For surfaces S of the second type we have, of course

$$c_1^2 c_2^2 + 4(c_1^3 + c_2^3) + 18 c_1 c_2 - 27 = 0. \tag{19}$$

The locus of the flecnode curves of the osculating ruled surfaces R_1 is again a surface of two sheets, each of which is projective to S itself, unless one of the three distinct roots α_1, α_2, α_3 of the characteristic equation satisfies the condition

$$\alpha^2 + c_1 = 0,$$

This, again, can happen only if

$$c_2^2 + 4c_1 = 0. \tag{30$_1$}$$

The conditions (19) and $(30)_1$ yield

$$c_1 = -\frac{9}{\sqrt[3]{16}}, \qquad c_2 = -3\sqrt[3]{2},$$

the other possibilities, involving imaginary cube roots of unity as factors being disposed of as in the preceding paragraph. The double root α_3 of the characteristic equation will be

$$\alpha_3 = \frac{2(c_1^2 + 3c_2)}{c_1 c_2 - 9} = \frac{9}{25\sqrt[3]{4}},$$

whence

$$\alpha_2^{3/2} = \pm\frac{27}{250}, \qquad k = \frac{\alpha^{3/2} - 1}{\alpha^{3/2} + 1} = -\frac{223}{277} \quad \text{or} \quad -\frac{277}{223}.$$

Substituting these values in (25) we find the following result. *Any surface of the second type whose osculating ruled surfaces have flecnode curves one of whose branches is a plane curve is a projective transformation of the surface*

$$Z = \log \frac{X^{[illegible]}}{Y^{[illegible]}} . \qquad (36)$$

The flecnode curve of an osculating ruled surface of a surface of the third type never has a plane branch, and both of its branches are projective transformations of the corresponding asymptotic curve of the surface itself.

If we use the four points

$$y, \quad z = y_u, \quad \rho = y_v, \quad \sigma = y_{uv}$$

as tetrahedron of reference, the equations of the osculating linear complexes of R_1 and R_2 reduce to

$$\begin{aligned} \omega_{13} - 2\omega_{34} + c_1\,\omega_{42} &= 0, \\ \omega_{12} - 2\omega_{42} - c_2\,\omega_{34} &= 0 \end{aligned} \qquad (37)$$

respectively, and these are special if and only if c_1 or c_2 is equal to zero. The two complexes are, of course, in involution.* The osculating linear complexes of the asymptotic curves Γ' ($v =$ const.) and Γ'' ($u =$ const.) are respectively

$$-\omega_{14} + \omega_{23} = 0, \qquad \omega_{14} + \omega_{23} = 0. \qquad (38)$$

As distinguished from the general theory, *any two of these four linear complexes are in involution.*

The osculating linear complexes of Γ' and Γ'' have in common a linear congruence. According to the general theory† one of the directrices d of this congruence (that of the first kind) lies in the tangent plane of the surface point considered, while the other d' (that of the second kind), passes through this point. Their equations are

$$x_1 = x_4 = 0 \quad \text{and} \quad x_2 = x_3 = 0$$

respectively, so that d coincides with $P_z P_\rho$ and d' with $P_y P_\sigma$.

Since we have a line d and a line d' associated with every point of the surface, we thus obtain two congruences, the *directrix congruences of the first and second kind.* The curves on S which correspond to the developables of these congruences are called the *directrix curves* of S, and are the same for both congruences. Their differential equation, in our case, reduces to‡

$$c_1\,du^2 - c_2\,dv^2 = 0.$$

---

* *Second Memoir*, p. 90.

† *Loc. cit.*, p. 95.

‡ *Loc. cit.*, pp. 116 and 119.

Thus, *the directrix curves of S form a conjugate system and are obtained by putting $\bar{u}$ = const. and $\bar{v}$ = const., where*

$$\bar{u} = \sqrt{c_1}\, u + \sqrt{c_2}\, v, \qquad \bar{v} = \sqrt{c_1}\, u - \sqrt{c_2}\, v, \tag{39}$$

if c_1 and c_2 are both different from zero. If one only of these quantities vanishes, the directrix curves of both families coincide with one of the two families of asymptotic lines. If c_1 and c_2 both vanish, the directrix curves are indeterminate.

If c_1 and c_2 are both different from zero, and if the surface S is of the first type, its parametric equations referred to its directrix curves will be

$$y_k = e^{\frac{1}{2}\left(\frac{\alpha_k}{\sqrt{c_1}} + \frac{\beta_k}{\sqrt{c_2}}\right)\bar{u} + \frac{1}{2}\left(\frac{\alpha_k}{\sqrt{c_1}} - \frac{\beta_k}{\sqrt{c_2}}\right)\bar{v}} \qquad (k = 1, 2, 3, 4). \tag{40}$$

Consequently *the directrix curves of S are also anharmonic.*

If the point P moves along the directrix curve $\bar{u}$ = const., the line d will describe a developable of the directrix congruence of the first kind. It will meet the cuspidal edge of this developable in the point

$$\lambda' z + \mu' \rho,$$

where*

$$\lambda' : \mu' = \sqrt{c_2} : \sqrt{c_1}.$$

Consequently, *if in the expressions*

$$\eta_k = \sqrt{c_2}\, z_k + \sqrt{c_1}\, \rho_k = (\alpha_k \sqrt{c_2} + \beta_k \sqrt{c_1})\, y_k \quad (k = 1, 2, 3, 4), \tag{41}$$

we put $\bar{u}$ = const., we obtain the cuspidal edge of the developable described by d when P moves along a curve $\bar{u}$ = const. of a surface of the first type. If we allow $\bar{u}$ to vary, we obtain the locus of these cuspidal edges, i. e., one of the sheets of the focal surface of the directrix congruence of the first kind. The other sheet will be given by

$$\zeta_k = (\alpha_k \sqrt{c_2} - \beta_k \sqrt{c_1})\, y_k \qquad (k = 1, 2, 3, 4). \tag{42}$$

In general, both of these sheets are surfaces which are projectively equivalent to S itself. The only exceptions to this statement will arise when at least one of the number pairs a_k, β_k satisfies the equation

$$c_2 \alpha_k^2 - c_1 \beta_k^2 = 0.$$

But α_k and β_k are solutions of (7). Therefore in this case we should also have

$$c_1 \alpha_k - c_2 \beta_k = 0.$$

We are assuming that c_1 and c_2 are both different from zero. The last two equations then imply

$$a_k^3 = \beta_k^3, \qquad c_1^3 = c_2^3,$$

* *Loc. cit.*, p. 119.

and, as we have seen in several instances, we may substitute for this later condition the simpler one

$$c_1 = c_2,$$

without loss of generality.

On the other hand, if $c_1 = c_2$, we find from (7) that

$$\alpha^2 - \beta^2 + 2(\beta - \alpha) = 0$$

or

$$(\alpha - \beta)(\alpha + \beta - 2) = 0.$$

Therefore two of the roots α_1, α_2, of the characteristic equation (8) will satisfy the quadratic

$$\alpha^2 + 2\alpha + c_1 = 0$$

and we shall have $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$. The other two roots α_3, α_4 will be solutions of

$$\alpha^2 + 2(2 - \alpha) + c_1 = 0$$

and we shall have

$$\beta_3 = 2 - \alpha_3, \qquad \beta_4 = 2 - \alpha_4.$$

The assumption $\beta_3 = \alpha_3$ would imply $\alpha_3 = 1$, and therefore $c_1 = c_2 = -3$. But in this case equation (8) would have three equal roots and the surface S would not be of the first type.

Therefore, *each of the two sheets of the focal surface of the directrix congruence of the first kind is a projective transformation of the surface* S *itself if* $c_1^3 \neq c_2^3$. *If* $c_1^3 = c_2^3 \neq 0$, *one of the sheets of the focal surface degenerates into a straight line. If* $c_1 = c_2 = 0$, *both sheets degenerate into coplanar straight lines and the directrix curves of* S *are indeterminate.**

The general properties of the congruence may be studied by means of a system of partial differential equations satisfied by the four pairs of functions (η_k, ζ_k).† This system consists of two partial differential equations of the first, and two of the second order. In order to obtain these equations, let us form $\partial\eta_k / \partial\bar{v}$ and $\partial\zeta_k / \partial\bar{u}$. We find

$$\frac{\partial \eta_k}{\partial \bar{v}} = \tfrac{1}{2}(\alpha_k \sqrt{c_2} + \beta_k \sqrt{c_1})\left(\frac{\alpha_k}{\sqrt{c_1}} - \frac{\beta_k}{\sqrt{c_2}}\right) y_k = \frac{\alpha_k^2 c_2 - \beta_k^2 c_1}{2\sqrt{c_1 c_2}} y_k,$$

or, making use of (7),

$$\frac{\partial \eta_k}{\partial \bar{v}} = \frac{c_1 \alpha_k - c_2 \beta_k}{\sqrt{c_1 c_2}} y_k.$$

* This theorem has here actually been proved only for surfaces of the first type. It may be shown to be true in the other cases also.

† E. J. Wilczynski, *Sur la théorie générale des congruences*, Mémoires publies par la Classe des Sciences de l'Académie royale de Belgique, collection en 4°, Deuxieme série, tome III (1911).

We have further, from (41) and (42),

$$2\alpha_k \sqrt{c_2}\, y_k = \eta_k + \zeta_k, \qquad 2\beta_k \sqrt{c_1}\, y_k = \eta_k - \zeta_k,$$

and therefore

$$\frac{\partial \eta_k}{\partial \bar{v}} = -\mu\eta_k + \lambda\zeta_k \qquad (k = 1, 2, 3, 4),$$

where

$$\lambda = \frac{c_2^{\frac{3}{2}} + c_1^{\frac{3}{2}}}{2c_1 c_2}, \qquad \mu = \frac{c_2^{\frac{3}{2}} - c_1^{\frac{3}{2}}}{2c_1 c_2}. \tag{43}$$

In precisely the same way we find the second equation of the following system:

$$\frac{\partial \eta_k}{\partial \bar{v}} = \frac{\partial \zeta_k}{\partial \bar{u}} = -\mu\eta_k + \lambda\zeta_k. \tag{44}$$

If we put

$$\eta = e^{-\mu\bar{v}}\mathfrak{y}, \qquad \zeta = e^{\lambda\bar{u}}\mathfrak{z}, \tag{45}$$

we find that $\mathfrak{y}_k$ and $\mathfrak{z}_k$ are simultaneous solutions of the equations

$$\frac{\partial \mathfrak{y}}{\partial \bar{v}} = m\mathfrak{z}, \qquad \frac{\partial \mathfrak{z}}{\partial \bar{u}} = n\mathfrak{y}, \tag{46}$$

where

$$m = \lambda e^{\lambda\bar{u}+\mu\bar{v}}, \qquad n = -\mu e^{-\lambda\bar{u}-\mu\bar{v}}. \tag{47}$$

The equations (46) are the required first order equations of our system.

In order to find the second order equations of our system, we observe in the first place that $\eta_1, \cdots \eta_4$, being linear homogeneous functions of $y_1, \cdots y_4$ with constant coefficients, satisfy the same system of partial differential equations as the latter functions, viz.:

$$\eta_{uu} + 2\eta_v + c_1\eta = 0, \qquad \eta_{vv} + 2\eta_u + c_2\eta = 0. \tag{48}$$

Since

$$\bar{u} = \sqrt{c_1}\, u + \sqrt{c_2}\, v, \qquad \bar{v} = \sqrt{c_1}\, u - \sqrt{c_2}\, v,$$

we find

$$\eta_u = \sqrt{c_1}\,(\eta_{\bar{u}} + \eta_{\bar{v}}), \qquad \eta_v = \sqrt{c_2}\,(\eta_{\bar{u}} - \eta_{\bar{v}}),$$

$$\eta_{uu} = c_1\,(\eta_{\bar{u}\bar{u}} + 2\eta_{\bar{u}\bar{v}} + \eta_{\bar{v}\bar{v}}),$$

$$\eta_{uv} = \sqrt{c_1 c_2}\,(\eta_{\bar{u}\bar{u}} - \eta_{\bar{v}\bar{v}}),$$

$$\eta_{vv} = c_2\,(\eta_{\bar{u}\bar{u}} - 2\eta_{\bar{u}\bar{v}} + \eta_{\bar{v}\bar{v}}).$$

Consequently we obtain for η as function of $\bar{u}$, $\bar{v}$ the following system of partial differential equations

$$\begin{aligned} &\eta_{\bar{u}\bar{u}} + \eta_{\bar{v}\bar{v}} + 2\lambda\eta_{\bar{u}} - 2\mu\eta_{\bar{v}} + \eta = 0, \\ &\eta_{\bar{u}\bar{v}} + \mu\eta_{\bar{u}} - \lambda\eta_{\bar{v}} = 0. \end{aligned} \tag{49}$$

Of course $\zeta_1, \cdots \zeta_4$ satisfy the same equations. The transformation (45) now yields the equations

$$
(50)\qquad
\begin{gathered}
\mathfrak{y}_{\bar u\bar u} + \mathfrak{y}_{\bar v\bar v} + 2\lambda\mathfrak{y}_{\bar u} - 4\mu\mathfrak{y}_{\bar v} + (1 + 3\mu^2)\,\mathfrak{y} = 0,\\
\mathfrak{y}_{\bar u\bar v} - \lambda\mathfrak{y}_{\bar v} + \lambda\mu\mathfrak{y} = 0,\\
\mathfrak{z}_{\bar u\bar u} + \mathfrak{z}_{\bar v\bar v} + 4\lambda\mathfrak{z}_{\bar u} - 2\mu\mathfrak{z}_{\bar v} + (1 + 3\lambda^2)\,\mathfrak{z} = 0,\\
\mathfrak{z}_{\bar u\bar v} + \mu\mathfrak{z}_{\bar u} + \lambda\mu\mathfrak{z} = 0.
\end{gathered}
$$

But from (46) we find

$$
\mathfrak{y}_{\bar v} = m\mathfrak{z}, \qquad \mathfrak{z}_{\bar u} = n\mathfrak{y},
$$

$$
\mathfrak{y}_{\bar v\bar v} = m_{\bar v}\,\mathfrak{z} + m\mathfrak{z}_{\bar v}, \qquad \mathfrak{z}_{\bar u\bar u} = n_{\bar u}\,\mathfrak{y} + n\mathfrak{y}_{\bar u}.
$$

so that, for the first and third equation of system (50), we may substitute the following:

$$
(51)\qquad
\begin{gathered}
\mathfrak{y}_{\bar u\bar u} = -(1 + 3\mu^2)\,\mathfrak{y} + 3\mu m\mathfrak{z} - 2\lambda\mathfrak{y}_{\bar u} - m\mathfrak{z}_{\bar v},\\
\mathfrak{z}_{\bar v\bar v} = -3\lambda n\mathfrak{y} - (1 + 3\lambda^2)\,\mathfrak{z} - n\mathfrak{y}_{\bar u} + 2\mu\mathfrak{z}_{\bar v}.
\end{gathered}
$$

Equations (46) *and* (51) *together constitute the system of partial differential equations characteristic of the directrix congruence of the first kind.* We have actually proved this for the case of a surface of the first type, but the statement remains true for surfaces of the second and third type as well.

The partial differential equations of the congruence show that *it is a W-congruence.** This also follows from the fact that the point correspondence between the two sheets of its focal surface, which is established by the lines of the congruence, is a projective transformation.

The line d, of the congruence, which belongs to the values u, v of the parameters, is obtained by joining the points P_z and P_ρ, where

$$
z = y_u, \qquad \rho = y_v.
$$

If the surface S is of the first type,

$$
\alpha_1 y_1, \quad \alpha_2 y_2, \quad \alpha_3 y_3, \quad \alpha_4 y_4
$$

and

$$
\beta_1 y_1, \quad \beta_2 y_2, \quad \beta_3 y_3, \quad \beta_4 y_4
$$

will, therefore, be the coördinates of two points of d. The Plückerian line coördinates of d will therefore be equal to

$$
\omega_{ik} = l_{ik}\, y_i\, y_k \qquad (i, k = 1, 2, 3, 4),
$$

where

$$
l_{ik} = \alpha_i \beta_k - \alpha_k \beta_i .
$$

* *Loc. cit.*, p. 56.

Consequently the coördinates of d will satisfy the quadratic equations

$$l_{13}\,l_{42}\,\omega_{12}\,\omega_{34} - l_{12}\,l_{34}\,\omega_{13}\,\omega_{42} = 0,$$

$$l_{13}\,l_{42}\,\omega_{14}\,\omega_{23} - l_{14}\,l_{23}\,\omega_{13}\,\omega_{42} = 0,$$

$$l_{14}\,l_{23}\,\omega_{12}\,\omega_{34} - l_{12}\,l_{34}\,\omega_{14}\,\omega_{23} = 0,$$

which, on account of the identical relations

$$\omega_{12}\,\omega_{34} + \omega_{13}\,\omega_{42} + \omega_{14}\,\omega_{23} = 0,$$

$$l_{12}\,l_{34} + l_{13}\,l_{42} + l_{14}\,l_{23} = 0,$$

are equivalent to a single one of them. Therefore, *the lines of the directrix congruence of the first kind, of a surface of the first type, belong to a tetrahedral complex whose fundamental tetrahedron coincides with the tetrahedron left invariant by the collineation group of the surface.* The invariant of this tetrahedral complex may be written in the form

$$\frac{l_{12}\,l_{34}}{l_{13}\,l_{42}}.$$

This theorem is modified for surfaces of the second type. *The group of collineations which leave invariant a surface of the second type has no invariant tetrahedron. The quadratic complex to which the directrix congruence of the first kind belongs in this case has the equation*

$$l_{13}\,l_{23}\,\omega_{12}\,\omega_{34} + l_{12}\,(\alpha_3^2 + \beta_3)\,\omega_{13}\,\omega_{23} = 0.$$

It consists of the lines which meet corresponding pairs of lines of two projective pencils, the plane of one (and only one) of these pencils passing through the vertex of the other.

The directrix congruence of the first kind, of a surface of the third type, does not belong to any non-degenerate quadratic complex. Its lines belong, instead, to a unique linear complex.

Let us now consider the directrix congruence of the second kind. If P describes one of the directrix curves on S, the directrix d' will describe one of the developables of this congruence, and will meet the cuspidal edge of this developable in the point

$$\lambda y + 2\mu\sigma,$$

where*

$$\frac{\lambda}{\mu} = -8 + 2c_2\frac{dv}{du}.$$

Thus, if P describes the curve $\bar{u} = \text{const.}$, d' generates a developable whose

* *Second Memoir*, p. 117.

cuspidal edge is given by

$$(52) \qquad 2\eta_k' = (-8 - 2\sqrt{c_1 c_2})\, y_k + 2\sigma_k = 2\,(\alpha_k \beta_k - 4 - \sqrt{c_1 c_2})\, y_k .$$

Similarly, the cuspidal edge of the developable described by d', when P moves along the curve $\bar{v} =$ const., is given by

$$(53) \qquad \zeta_k' = (\alpha_k \beta_k - 4 + \sqrt{c_1 c_2})\, y_k \qquad (k = 1, 2, 3, 4).$$

The surfaces $S_{\eta'}$ and $S_{\zeta'}$ constitute the focal surface of the congruence. They are projective transformations of S itself and of each other, unless at least one pair of the quantities α_k, β_k satisfies the equation

$$(\alpha\beta - 4)^2 = c_1 c_2,$$

which according to (7) is equivalent to

$$4\alpha\beta - 2\,(c_1 \alpha + c_2 \beta) - 16 = 0.$$

But this equation, together with (7), implies $c_1^3 = c_2^3$. Thus *the reduction of one of the sheets of the focal surface to a straight line takes place for both directrix congruences simultaneously if* $c_1^3 = c_2^3$, *and in no other case.*

We find

$$\frac{\partial \eta_k'}{\partial \bar{v}} = \tfrac{1}{2}\,(\alpha_k \beta_k - 4 - \sqrt{c_1 c_2}) \left(\frac{\alpha_k}{\sqrt{c_1}} - \frac{\beta_k}{\sqrt{c_2}} \right) y_k = \frac{c_2^{3/2} - c_1^{3/2}}{\sqrt{c_1 c_2}}\, y_k ,$$

$$\frac{\partial \zeta_k'}{\partial \bar{u}} = \tfrac{1}{2}\,(\alpha_k \beta_k - 4 + \sqrt{c_1 c_2}) \left(\frac{\alpha_k}{\sqrt{c_1}} + \frac{\beta_k}{\sqrt{c_2}} \right) y_k = \frac{c_2^{3/2} + c_1^{3/2}}{\sqrt{c_1 c_2}}\, y_k ,$$

and

$$\zeta_k' - \eta_k' = 2\sqrt{c_1 c_2}\, y_k ,$$

so that we have

$$\frac{\partial \eta_k'}{\partial \bar{v}} = \mu\,(\zeta_k' - \eta_k'), \qquad \frac{\partial \zeta_k'}{\partial \bar{u}} = \lambda\,(\zeta_k' - \eta_k'),$$

making use of the notations (43). Let us assume λ and μ both different from zero, and put

$$(54) \qquad \eta_k' = \mu e^{-\mu \bar{v}}\, \mathfrak{y}_k', \qquad \zeta_k' = \lambda e^{\lambda \bar{u}}\, \mathfrak{z}_k',$$

and we shall find

$$(55) \qquad \frac{\partial \mathfrak{y}_k'}{\partial \bar{v}} = m \mathfrak{z}_k', \qquad \frac{\partial \mathfrak{z}_k'}{\partial \bar{u}} = n \mathfrak{y}_k',$$

where m and n have the same values as in (47).

On account of (52) and (53), η_k' and ζ_k' satisfy the same system of partial differential equations (6) of which $y_1, \cdots y_4$ are solutions. If we introduce $\bar{u}$ and $\bar{v}$ as independent variables, we find that both $\eta_1', \cdots \eta_4'$ and $\zeta_1', \cdots \zeta_4'$ are solutions of (49). Since the transformations (54) differ from (45) merely by constant factors, we find that $\mathfrak{y}'_k$ and $\mathfrak{z}'_k$ satisfy equations (50), so that we shall have in particular

$$\mathfrak{y}'_{\bar{u}\bar{u}} = -\mathfrak{y}'_{\bar{v}\bar{v}} - 2\lambda\mathfrak{y}'_{\bar{u}} + 4\mu\mathfrak{y}'_{\bar{v}} - (1 + 3\mu^2)\,\mathfrak{y}',$$

$$\mathfrak{z}'_{\bar{v}\bar{v}} = -\mathfrak{z}'_{\bar{u}\bar{u}} - 4\lambda\mathfrak{z}'_{\bar{u}} + 2\mu\mathfrak{z}'_{\bar{v}} - (1 + 3\lambda^2)\,\mathfrak{z}'.$$

If we make use of (55), we find

(56)
$$\mathfrak{y}_{\bar{u}\bar{u}} = -(1 + 3\mu^2)\,\mathfrak{y}' + 3\mu m\mathfrak{z}' - 2\lambda\mathfrak{y}'_{\bar{u}} - m\mathfrak{z}'_{\bar{v}},$$
$$\mathfrak{z}'_{\bar{v}\bar{v}} = -3\lambda n\mathfrak{y}' - (1 + 3\lambda^2)\,\mathfrak{z}' - n\mathfrak{y}'_{\bar{u}} + 2\mu\mathfrak{z}'_{\bar{v}}.$$

Equations (55) and (56) are precisely the same as (46) and (51) except for the letters used for the unknown functions. Consequently *the two directrix congruences are equivalent to each other by means of a projective transformation.* If λ or μ should vanish, the transformation (54) ceases to be available, but the two congruences are nevertheless projectively equivalent. Since, in this case, we may assume $c_1 = c_2 = c$, the discriminant Δ becomes equal to

$$c^4 + 8c^3 + 18c^2 - 27 = (c + 3)^3\,(c - 1).$$

The surface will be of the first type only if c is neither unity nor minus three.

THE UNIVERSITY OF CHICAGO,
August 27, 1912.